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Symbolic Treatment of Exact Linear Differential Equations.

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1. The linear equation is here supposed to consist of terms of the form

$$Ax^s \frac{d^r y}{dx^r}, \text{ or } Ax^s D^r y,$$

in which r is zero or a positive integer, and s is unrestricted. Let these terms be grouped together in such a way that the value of $s - r$ is the same for all the terms in a group; then, if m be the least value of r for the terms in a group, and $q - m = s - r$, the group may be written

$$x^q \left[A_0 + A_1 xD + A_2 x^2 D^2 + \ldots.\right] D^m y. \tag{1}$$

Using ϑ to denote the operator $x\dfrac{d}{dx}$ or xD, we may reduce this expression by means of the theorem

$$x^n D^n = \vartheta(\vartheta - 1)(\vartheta - 2) \ldots. (\vartheta - n + 1), \tag{A}$$

to the form

$$x^q f(\vartheta) D^m y. \tag{2}$$

2. In order that the differential equation may be exact, each group of the form (2) contained in it must separately constitute an exact derivative; and the process of direct integration is equivalent to that of resolving, in the case of each group, the symbolic operator into factors of which that most remote from the operand is the simple factor D. It is well known that if m is not zero and q is an integer less than m, every term in (1), and therefore the group itself, is an exact derivative. The symbolic transformation of the expression (2) may in this case be effected by the formula deduced below.

3. We have by differentiation

$$D.xDy = xD^2y + Dy,$$

or

$$D\vartheta y = \vartheta Dy + Dy;$$

whence, symbolically,

$$\vartheta D = D(\vartheta - 1). \tag{3}$$

Operating upon both members of this equation with ϑ, we have

$$\vartheta^2 D = \vartheta D(\vartheta - 1) = D(\vartheta - 1)^2,$$

and in like manner we find generally

$$\vartheta^r D = D(\vartheta - 1)^r;$$

whence, f denoting a rational integral function,

$$f(\vartheta)D = Df(\vartheta - 1). \tag{4}$$

Again, operating with each member of this equation upon Dy, we have

$$f(\vartheta)D^2 = Df(\vartheta - 1)D = D^2 f(\vartheta - 2),$$

and in like manner,

$$f(\vartheta)D^3 = D^2 f(\vartheta - 2)D = D^3 f(\vartheta - 3),$$

and generally,

$$f(\vartheta)D^m = D^m f(\vartheta - m). \tag{B}$$

4. Applying this formula, in the case mentioned above, in which $m - q$ is a positive integer, to the expression (2), we have

$$x^q f(\vartheta)D^m = x^q D^m f(\vartheta - m) = x^q D^q D^{m-q} f(\vartheta - m).$$

By the formula (A) this becomes

$$x^q f(\vartheta)D^m = \vartheta(\vartheta - 1) \ldots . (\vartheta - q + 1)D^{m-q} f(\vartheta - m),$$

and, making a second application of formula (B), we have

$$x^q f(\vartheta)D^m = D^{m-q}(\vartheta - m + q) \ldots . (\vartheta - m + 1)f(\vartheta - m), \tag{5}$$

in which a resolution of the operator into symbolic factors of the form required is effected.

5. Formula (A), by which we reduce the groups to the form (2), is also readily deduced from equation (3); for, multiplying the latter by x, we have

$$x^2 D^2 = \vartheta(\vartheta - 1);$$

and operating with each member of this equation upon Dy, we find

$$x^2 D^3 = \vartheta(\vartheta - 1)D,$$

which, by equation (4), becomes

$$x^2 D^2 = D(\vartheta - 1)(\vartheta - 2),$$

and multiplying by x,

$$x^3 D^3 = \vartheta(\vartheta - 1)(\vartheta - 2).$$

Thus we may successively derive all the results included in equation (A).

6. When m is zero, and also when m is an integer, unless q is zero or a positive integer less than m, the possibility of resolving the operator into factors of the required form depends upon the existence of a proper factor in $f(\vartheta)$.

By differentiation we have

$$Dx^{q+1}y = x^{q+1}Dy + (q + 1)x^q y;$$

whence, using Dx^{q+1} as a symbol of operation,

$$x^q(\vartheta + q + 1) = Dx^{q+1}. \tag{C}$$

Hence, if $\vartheta + q + 1$ is a factor of $f(\vartheta)$, that is, if $-(q+1)$ is a root of $f(\vartheta) = 0$, so that

$$f(\vartheta) = (\vartheta + q + 1)\phi(\vartheta), \tag{6}$$

ϕ being rational and integral, then the expression (2) may be transformed thus,

$$x^q f(\vartheta) D^m y = Dx^{q+1}\phi(\vartheta) D^m y, \tag{7}$$

in which, subject to the condition (6), the operator is resolved into factors of the required form.

7. This condition includes that mentioned in (2); for, if we transform the operator by formulae (B) and (A) thus,

$$\begin{aligned} x^q f(\vartheta) D^m &= x^{q-m} x^m D^m f(\vartheta - m) \\ &= x^{q-m}\vartheta(\vartheta - 1) \ldots . (\vartheta - m + 1) f(\vartheta - m), \end{aligned} \tag{8}$$

the condition of direct integrability is the occurrence of the factor $\vartheta + q - m + 1$. Now if q is an integer less than m, this factor occurs among those actually written in equation (8); thus the condition is in that case satisfied independently of the form of f. Otherwise, the factor can only occur as a factor of $f(\vartheta - m)$, and the condition of direct integrability then is that $f(\vartheta - m)$ shall vanish with $\vartheta + q - m + 1$; that is, that $f(-q-1) = 0$, as before. Thus nothing is gained by the transformation (8), and it is better to employ the transformation (5) when, m being a positive integer, q is zero or an integer less than m, and the transformation (7), when, this condition not being fulfilled, $f(\vartheta)$ satisfies the condition (6).

8. The criteria thus established show at once by what powers of x the group being multiplied becomes exact, and thus, in an equation containing two or more groups, whether there be an integrating factor of the form x^p. For example, the equation

$$2x^2(x+1)\frac{d^2y}{dx^2} + x(7x+3)\frac{dy}{dx} - 3y = X$$

contains two groups, which being written in the form (2), the equation is

$$x^2(2\vartheta + 7)Dy + (2\vartheta + 3)(\vartheta - 1)y = X.$$

The first term becomes exact according to the first condition when multiplied by x^{-2}, and exact according to the second condition when multiplied by $x^{\frac{1}{2}}$. The second term becomes exact in this case when multiplied by either of the same factors. Hence we may write the given operator

$$2x^2(x+1)D^2 + x(7x+3)D - 3$$

in either of the forms

$$x^2 D\left[(2\vartheta+5)+x^{-1}(2\vartheta+3)\right],$$

or

$$2x^{-\frac{1}{2}} D\left[x^{\frac{7}{2}} D+x^{\frac{3}{2}}(\vartheta-1)\right].$$

Again, in the first of these expressions, each of the two terms between the brackets fulfills the second condition when multiplied by $x^{\frac{3}{2}}$; the equation may therefore be further reduced to

$$2x^2 D x^{-\frac{3}{2}} D\left(x^{\frac{5}{2}}+x^{\frac{3}{2}}\right) y=X.$$

The value of y obtained by performing on X the inverse operations in their proper order is

$$y=\frac{1}{2x^{\frac{3}{2}}(x+1)} \int x^{\frac{3}{2}} \int x^{-2} X dx dx.$$

In like manner the equation might be reduced to

$$2x^{-\frac{1}{2}} D x^{\frac{7}{2}} D\left(1+x^{-1}\right) y=X,$$

giving

$$y=\frac{x}{2(x+1)} \int x^{-\frac{7}{2}} \int x^{\frac{1}{2}} X dx dx;$$

but perhaps the best expression for y is that which results from elimination between the two first integrals, namely,

$$y=\frac{x}{5(x+1)} \int x^{-2} X dx-\frac{1}{5x^{\frac{3}{2}}(x+1)} \int x^{\frac{1}{2}} X dx.$$

It is noticeable that, whenever an equation of the form considered is susceptible of two successive direct integrations with intermediate multiplication by a power of x, it is also susceptible of direct integration when multiplied by either of two different powers of x.

9. As a second example, let us take the equation

$$\left(x^3-x\right) \frac{d^3 y}{dx^3}+\left(8x^2-3\right) \frac{d^2 y}{dx^2}+14x \frac{dy}{dx}+4y=0,$$

which, when reduced by formula (A), becomes

$$(\vartheta+1)(\vartheta+2)^2 y-(\vartheta+3) D^2 y=0.$$

Reducing the first term by a double application of formula (C), and the second by formula (B), the equation becomes

$$D^2\left[\left(x^3-x\right) D+2x^2-1\right] y=0.$$

In this case there is no intermediate x-factor between the D's, and the symbolic operator

$$\left(x^3-x\right) D^3+\left(8x^2-3\right) D^2+14xD+4$$

does not admit of resolution into factors of the first order in two separate ways.

10. As a final example, consider the equation

$$(2x^2 + x^{\frac{3}{2}})\frac{d^3y}{dx^3} + 3x\frac{d^2y}{dx^2} - 3\frac{dy}{dx} = X,$$

which becomes $$(2\vartheta + 3)(\vartheta - 1)\,Dy + x^{\frac{1}{2}}\vartheta D^2y = X.$$

The first term becomes exact by the second condition when multiplied by $x^{\frac{1}{2}}$, and the second term will then be exact by the first condition. The first member of the equation thus becomes

$$x^{-\frac{1}{2}}D\,[2x^{\frac{3}{2}}(\vartheta - 1)\,D + (\vartheta - 1)(\vartheta - 2)]\,y,$$

in which $\vartheta - 1$ is not a factor since it is not commutative with D; but, transforming the first term by formula (B), the expression becomes

$$x^{-\frac{1}{2}}D\,[(2x^{\frac{3}{2}} + x)\,D - 1]\,(xD - 2)\,y,$$

in which the operator is resolved into three factors of the first order.

In the original expression we might, however, have reserved the factor D which stands nearest to the operand, thus,

$$[(2\vartheta + 3)(\vartheta - 1) + x^{\frac{1}{2}}\vartheta D]\,Dy = X.$$

The factor in parenthesis cannot now be made exact; but, transforming the second term by formula (B), the first member may be written

$$[(2x + x^{\frac{1}{2}})\,D + 3]\,(xD - 1)\,Dy,$$

in which a second resolution of the operator into factors of the first order has been effected.